Futo Journal Series (FUTOJNLS)
e-ISSN :2476-8456 p-ISSN : 2467-8325
Volume-2, Issue-1, pp- 178 - 185
www.futojnls.org

Research Paper July 2016

Application of Taylor's Series in Ritz Method for Free Vibration Analysis of Simply Supported Rectangular Thin Orthotropic Plate

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Abstract

In this article, Taylor's series was applied in Ritz method for the purpose of performing free vibration analysis of a simply supported rectangular thin orthotropic plate. Taylor's series was used to develop the general plate deflection function of a simply supported rectangular thin orthotropic plate, which was truncated at the fourth term. The natural frequency equation of the plate was determined and used for computing the natural frequencies of the simply supported plate with different aspect ratios, p, and different flexural rigidity ratios, φ. The values of the natural frequencies obtained for the simply supported rectangular thin orthotropic plate vibrating in the first mode ranged from 11.172 radians per second to 996.833 radians per second for different aspect ratios (ranging from 0.1 to 2 in steps of 0.1) and different flexural rigidities. The average percentage difference in the values of the natural frequency with exact solutions for different flexural rigidity ratios are 0.0458%, 0.0564%, and 0.0528% when aspect ratio, p, is equal to b/a. Thus, the results obtained from the present study closely agree with those for the exact deflection function of free vibrating rectangular thin orthotropic plate.

Keywords - *Orthotropic plate, Rayleigh-Ritz method, Rectangular plate, Taylor's series, Vibration*

1.0 Introduction

Different theories are available to handle vibration problems of rectangular plates, and correspondingly, several new methods have been developed to analyze these problems. The direct integration method (equilibrium approach) gives exact solutions, but it is only limited to analyzing statically loaded simply supported rectangular plates on all four edges (i.e. SSSS plate) due to difficulty encountered in using the method in the formulation of shape functions. For example, it is worthy to note that for centuries, exact solutions of a fully clamped rectangular plate (i.e. CCCC plate) have not been obtained, and it is widely believed that an exact solution is not achievable for analyzing CCCC plate. The energy approach such as the Ritz method, which is a generalized version of the Rayleigh method, can be applied to the solution of either statically or dynamically loaded SSSS plate if polynomial series are employed in place of trigonometric series. This implies that approximate continuum or numerical methods must be resorted to for solution of plates. Although numerical approaches such as the finite element method and the finite difference method have wider applications to numerous boundary conditions, they are tedious and require extra computational efforts. Various authors employed various approaches in the free vibration analysis of thin rectangular plates. Srinivas and Rao (1970) and Srinivas *et al*. (1970) analyzed free vibration of homogeneous isotropic and orthotropic simply supported rectangular plates. Hsu (2003) analyzed the vibration of isotropic and orthotropic plates with mixed boundary conditions using the differential quadrature method. Pilkey (2005) derived natural frequency equations for the free vibration of rectangular orthotropic plates of various boundary conditions using the exact method. Wu *et al*. (2007) found exact solutions for free vibration analysis of rectangular plates with various edge conditions using Bessel functions. Xing and Liu (2009) provided exact solutions for free vibration of thin orthotropic rectangular plates by using a novel separation of variables.

Many solutions to vibration problems in literature such as Leissa (1973) and Meirovitch (1997) employed trigonometric series formulated shape functions. It is possible to use trigonometric series to formulate the shape function of the SSSS plate, CCCC plate, or a plate with two opposite clamped edges and the other two opposite edges simply supported (i.e. CSCS plate) (Ugural, 1999; Ventsel & Krauthammer, 2001). However, it cannot be applied if the opposite edges of the rectangular plate are different (for example, if two opposite edges of the rectangular plate are simply supported and clamped). The shape functions of many rectangular plates such as plates simply supported on the first edge and clamped on the other three edges (i.e. SCCC plate), clamped on one edge and simply supported on the other three edges (i.e. CSSS plate), simply supported on two adjacent edges and clamped and free on the other edges (i.e. SSCF plate), and simply supported on opposite sides and clamped and free on the other edges (i.e. SCSF plate), cannot be formulated using trigonometric series because it involves great deal of analysis and time. Many other boundary conditions limit the use of trigonometric series (Ugural, 1999; Ventsel & Krauthammer, 2001). In view of these obvious constraints associated with formulating the shape functions of rectangular plates of various boundary conditions by trigonometric series, another method of developing a deflection function as proposed by Ibearugbulem (2012) is applied in this paper. The Taylor's series in Rayleigh–Ritz method is used herein in formulating the shape function so as to overcome the deficiencies of the trigonometric series, and to arrive at very close approximations of the exact solutions with reduced computation efforts. The advantages of Taylor's series shape function lie in the ease of application and versatility to numerous plate boundary conditions.

In this paper, the authors present a solution to the free vibration analysis of a simply supported rectangular thin orthotropic plate. The shape function which satisfied the plate boundary conditions is approximated using Taylor's series truncated at the fourth term. The total potential energy functional by Rayleigh–Ritz method as derived by Abamara (2014) is presented. Values of the natural frequency are obtained for different aspect ratios and different flexural rigidity ratios, and the results are compared with exact solutions derived by Pilkey (2005).

2.0 Theoretical Background

Ventsel and Krauthammer (2001) gave the fourth-order homogenous partial differential equation for the undamped, free, linear vibration of a plate as:

$$D\nabla^4 w(x,y,t) + \frac{\rho h \partial^2 w}{\partial t^2}(x,y,t) = 0 \quad (1)$$

where D is the flexural rigidity of the plate, ∇^4 is the biharmonic differential operator, $w(x,y,t)$ is the deflection of the plate, x and y are the co-ordinate axes of the plate in the horizontal and vertical directions respectively, t is time, ρ is the mass density of the material, and h is the plate thickness. The biharmonic operator in (1) is expressed as given in (2).

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \quad (2)$$

Ibearugbulem (2012) derived the following shape function for a thin rectangular plate using Taylor – Maclaurin's series as:

$$w = w(x,y) = \sum\sum \frac{F^{(m)}(x_0).F^{(n)}(y_0)}{m!\,n!}.(x-x_0)^m.(y-y_0)^n \quad (3)$$

where $F^{(m)}(x_0)$ is the m^{th} partial derivative of the function w with respect to y. The factorials $m!$ and $n!$ are factorials of m and n respectively while x_0 and y_0 are the points at the origin, which is taken at zero. By truncating the infinite series in (3) at $m = n = 4$, he obtained (4):

$$w = \sum_{m=0}^{4} \sum_{n=0}^{4} J_m K_n R^m Q^n \tag{4}$$

where J_m and K_n are expressed as given in (5a) and (5b).

$$J_m = \frac{a^m . F^m(0)}{m!} \tag{5a}$$

$$K_n = \frac{b^n . F^{(n)}(0)}{n!} \tag{5b}$$

The terms, R and Q, in (4) are dimensionless parameters given as:

$$R = X/a; \quad Q = Y/b \tag{6}$$

When $m = n = 4$, the function given in (4) can be expanded in the form:

$$w(R,Q) = (J_0 + J_1 R + J_2 R^2 + J_3 R^3 + J_4 R^4)(K_0 + K_1 Q + K_2 Q^2 + K_3 Q^3 + K_4 Q^4) \tag{7}$$

where J_i and K_i (i = 0, 1, 2, 3, 4) are unknown constants of the polynomial series deflection function.

In order to obtain the deflection function, the boundary conditions of the all-round simply supported (SSSS) rectangular thin orthotropic plate, have to be taken into consideration. The boundary conditions of the SSSS plate shown in Fig. 1 are as follows:

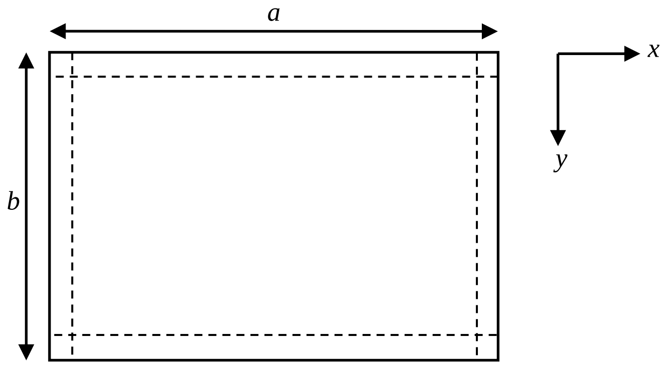

Figure 1. Rectangular plate with all-round simply supported boundary conditions

$$w(R = 0) = w''^{R}(R = 0) = 0 \tag{8}$$

$$w(R = 1) = w''^{R}(R = 1) = 0 \tag{9}$$

$$w(Q = 0) = w''^{Q}(Q = 0) = 0 \tag{10}$$

$$w(Q = 1) = w''^{Q}(Q = 1) = 0 \tag{11}$$

Substituting (8) and (10) into (7), yields:

$J_0 = 0;\ J_2 = 0;\ K_0 = 0;\ K_2 = 0$

Also, substituting (9) into (7) and solving them simultaneously gives:

$J_1 = J_4;\ J_3 = -2J_4$

Similarly, substituting (11) into (7), and solving the resulting two equations simultaneously, yields:

$K_1 = K_4;\ K_3 = -2K_4$

Substituting the values of J_i and K_i (where i = 0, 1, …, 4) into the (3), yields:

$$w = (R - 2R^3 - R^4)(Q - 2Q^3 - Q^4)J_4 K_4 \tag{12}$$

Equation (12) can be re-written in the form:

$$w = AH \tag{13}$$

where A is the amplitude of the shape function and H is the buckling curve expression. From (12) and (13), we have:

$$A = J_4 K_4 \tag{14a}$$

$$H = (R - 2R^3 - R^4)(Q - 2Q^3 - Q^4) \tag{14b}$$

The partial derivatives of (12) with respect to R, Q, or both, are as follows:

$$w'^R = \frac{\partial w(R,Q)}{\partial R} = A(1 - 6R^2 + 4R^3)(Q - 2Q^3 + Q^4) \quad (15)$$

$$w''^R = \frac{\partial^2 w(R,Q)}{\partial R^2} = A(12R^2 - 12R)(Q - 2Q^3 + Q) \quad (16)$$

$$w'^Q = \frac{\partial w(R,Q)}{\partial Q} = A(R - 2R^3 + R^4)(1 - 6Q^2 + 4Q^3) \quad (17)$$

$$w''^Q = \frac{\partial^2 w(R,Q)}{\partial Q^2} = A(R^2 - 2R^2 + R^4)(12Q^2 - 12Q) \quad (18)$$

$$w''^{RQ} = \frac{\partial^2 w(R,Q)}{\partial R \partial Q} = A(1 - 6R^2 + 4R^3)(1 - 6Q^2 + 4Q^3) \quad (19)$$

Integrating the square of these equations given from (15) to (19) partially with respect to R and Q in a closed domain yields:

$$\int_0^1 \int_0^1 (w''^R)^2 \, \partial R \partial Q = A^2(4.8)(0.04921) = 0.23621A^2 \quad (20)$$

$$\int_0^1 \int_0^1 (w''^Q)^2 \, \partial R \partial Q = A^2(0.04921)(4.8) = 0.23621A^2 \quad (21)$$

$$\int_0^1 \int_0^1 (w''^{RQ}) \, \partial R \partial Q = A^2(0.48571)(0.048571) = 0.2351A^2 \quad (22)$$

$$\int_0^1 \int_0^1 (w)^2 = A^2(0.4920635)(0.4920635) = 0.00242127A^2 \quad (23)$$

Using the deflection function i.e. (12), the total potential energy, Π, for an all-round simply supported rectangular orthotropic plate experiencing free vibration is expressed as:

$$\Pi = \frac{D_x A^2}{2b^2} \int_0^1 \int_0^1 \left[\frac{\varphi_1}{p^3} \left(\frac{\partial^2 H}{\partial R^2} \right)^2 + 2\frac{\varphi_2}{p} \left(\frac{\partial^2 H}{\partial R \partial Q} \right)^2 + \rho \varphi_3 \left(\frac{\partial^2 H}{\partial Q^2} \right)^2 \right] \partial R \partial Q - \frac{pA^2 b^2 \lambda^2 \rho t}{2} \int_0^1 \int_0^1 H^2 \partial R \partial Q \quad (24)$$

where D_x is flexural rigidity in the x-direction, A is the amplitude of the shape function as defined in (14a), a and b are the plate dimensions in the x- and y-directions respectively, φ_i are ratios of the bending rigidities, R and Q are dimensionless parameters, λ is the natural frequency, t is the plate thickness, and H is the buckling curve expression as defined in (14b).

3.0 Derivation of Natural Frequency Equations

The natural frequency, λ, of a vibrating plate, is an important parameter used in the design of a vibrating structure. In this work, the natural frequency, λ, is obtained by minimizing the total potential energy equation and equating it to zero. After minimizing and equating (24) to zero, the square of natural frequency, λ^2, is obtained in terms of the aspect ratio, p, and dimension, b, as given in (25) for aspect ratio, $p = a/b$.

$$\lambda^2 = \frac{\frac{D_x}{b^4 \rho t} \int_0^1 \int_0^1 \left[\frac{\varphi_1}{p} \left(\frac{\partial^2 H}{\partial R^2} \right)^2 + \frac{2\varphi_2}{p^2} \left(\frac{\partial^2 H}{\partial R \partial Q} \right)^2 + \varphi_3 \left(\frac{\partial^2 H}{\partial Q^2} \right)^2 \right] \partial R \partial Q}{\int_0^1 \int_0^1 H^2 \partial R \partial Q} \quad (25)$$

In terms of the plate dimensions, a and b, the equation becomes:

$$\lambda^2 = \frac{\frac{D_x}{a^4 \rho t} \int_0^1 \left[\varphi_1 \left(\frac{\partial^2 H}{\partial R^2} \right)^2 + \frac{2\varphi_2 a^2 \left(\frac{\partial^2 H}{\partial R \partial Q} \right)^2}{b^2} + \frac{\varphi_3 a^4 \left(\frac{\partial^2 H}{\partial Q^2} \right)^2}{b^2} \right] \partial R \partial Q}{\int_0^1 \int_0^1 H^2 \partial R \partial Q} \quad (26)$$

and in terms of the aspect ratio, p, and dimension, a, the square of the natural frequency is given as:

$$\lambda^2 = \frac{\frac{D_x}{a^4 \rho t} \int_0^1 \int_0^1 \left[\varphi_1 \left(\frac{\partial^2 H}{\partial R^2} \right)^2 + 2\varphi_2 p^2 \left(\frac{\partial^2 H}{\partial R \partial Q} \right) + \varphi_3 p^4 \left(\frac{\partial^2 H}{\partial Q^2} \right) \right] \partial R \partial Q}{\int_0^1 \int_0^1 H^2 \partial R \partial Q} \quad (27)$$

Similarly, for the aspect ratio, $p = b/a$, the square of the natural frequency (i.e. λ^2), is given in terms of a and p as:

$$\lambda^2 = \frac{\frac{D_x}{a^4\rho t}\int_0^1\int_0^1\left[\varphi_1\left(\frac{\partial^2 H}{\partial R^2}\right)^2 + \frac{\left(\frac{\partial^2 H}{\partial R\partial Q}\right)^2}{p^2} + \varphi_3 p^4\frac{\left(\frac{\partial^2 H}{\partial Q^2}\right)^2}{p^4}\right]\partial R\partial Q}{\int_0^1\int_0^1 H^2\partial R\partial Q} \quad (28)$$

The natural frequency for aspect ratio, $p = b/a$ by substituting (15) – (18) into (20) and simplifying, yields the square of the natural frequency in terms of p and a as:

$$\lambda^2 = \frac{D_x}{a^4\rho t}[97.5562\varphi_1 + 194.8647p^2\varphi_2 + 97.5562p^4\varphi_3] \quad (29)$$

And for λ^2 in terms of the dimensions a and b, the ratio a/b is substituted in place of p in (26) and simplified.

$$\lambda^2 = \frac{D_x}{a^4\rho t}\left[97.5562\varphi_1 + 194.8647\frac{a^2}{b^2}\varphi_2 + 97.5562\frac{a^4}{b^4}\varphi_3\right] \quad (30)$$

For the reciprocal of the aspect ratio (i.e. $p = b/a$), the square of the natural frequency, λ, is given by:

$$\lambda^2 = \frac{D_x}{a^4\rho t}\left[97.5562\varphi_1 + 194.864\frac{\varphi_2}{p^2} + 97.5562\frac{\varphi_3}{p^4}\right] \quad (31)$$

The final equation derived in this work for the computation of the natural frequency, λ, of an all-round simply supported rectangular orthotropic thin plate in free vibration is given by (32) for aspect ratio, $p = a/b$ or by (33) for aspect ratio, $p = b/a$.

$$\lambda = \left\{\frac{D_x}{a^4\rho t}\left[97.5562\varphi_1 + 194.8647\left({}^{a}/_{b}\right)^2\varphi_2 + 97.5562\left({}^{a}/_{b}\right)^4\varphi_3\right]\right\}^{\frac{1}{2}} \quad (32)$$

$$\lambda = \left\{\frac{D_x}{a^4\rho t}\left[97.5562\varphi_1 + 194.8647\frac{\varphi_2}{p^2} + 97.5562\frac{\varphi_3}{p^4}\right]\right\}^{\frac{1}{2}} \quad (33)$$

Pilkey (2005) obtained the frequency equations of SSSS plate in vibration for $p = a/b$ and $p = b/a$, respectively, as given in (34) and (35).

$$\lambda^2 = \frac{1}{b^4\bar{m}}\left(\frac{97.409D_x}{p^4} + \frac{194.8647B}{p^2} + 97.409D_y\right) \quad (34)$$

$$\lambda^2 = \frac{1}{a^4\bar{m}}\left(97.409D_x + \frac{194.8647B}{\alpha^2} + \frac{97.409D_y}{\alpha^4}\right) \quad (35)$$

From (34) and (35), the equations of λ are:

$$\lambda = \left[\frac{1}{b^4\bar{m}}\left(\frac{97.409D_x}{p^4} + \frac{194.8647B}{p^2} + 97.409D_y\right)\right]^{\frac{1}{2}} \quad (36)$$

$$\lambda = \left[\frac{1}{a^4\bar{m}}\left(97.409D_x + \frac{194.8647B}{\alpha^2} + \frac{97.409D_y}{\alpha^4}\right)\right]^{\frac{1}{2}} \quad (37)$$

4.0 Results and Discussion

The natural frequencies, λ, were computed for different aspect ratios, $p = b/a$, and flexural rigidities, φ_i, and the values are given in Tables 1 – 3. The values of the frequency, λ of the present study for aspect ratio, p (ranging from 0.2 to 1.0) given in these tables were plotted as shown in Fig. 2. From the graph, it can be seen that for the three curves, the natural frequency decreased with increased aspect ratio, p. The natural frequency, however, decreased with decreasing flexural rigidity ratios. These results were compared in Tables 1 – 3 with the results obtained from the two equations i.e. (36) and (37) derived by Pilkey (2005) for all-round simply supported rectangular orthotropic thin plate.

The maximum value of the average percentage difference between the natural frequencies, λ, obtained in this work and those obtained by Pilkey (2005) was 0.0564 per cent. Thus, the natural frequencies obtained in this work closely approximate those of Pilkey (2005).

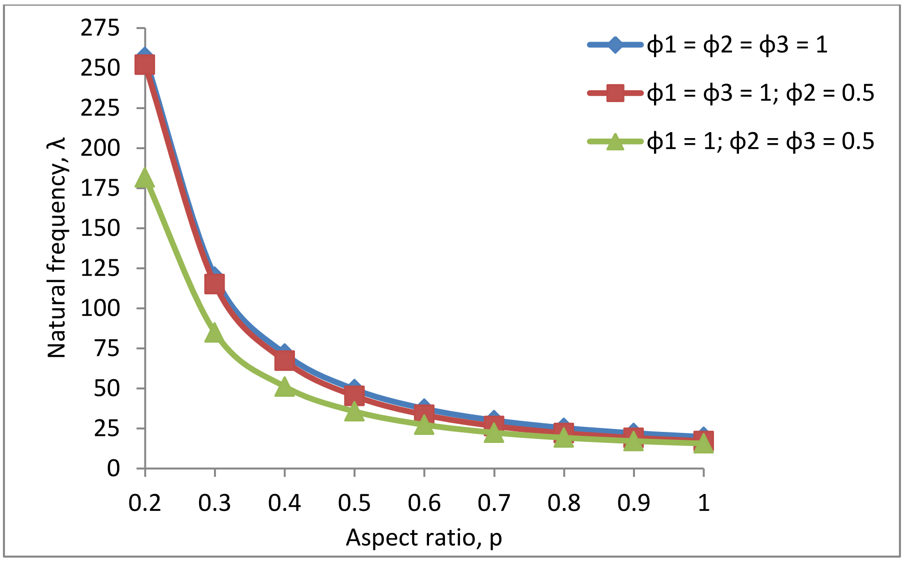

Figure 2. Influence of aspect ratio, *p*, and flexural rigidity ratio, φ, on natural frequency, λ

Table 1. Values of natural frequency, λ, for $\varphi_1 = \varphi_2 = \varphi_3 = 1$, $p = b/a$

p	λ (radians per second)		Difference	Percentage difference
	Present study	Pilkey (2005)		
0.1	997.570	996.833	0.737	0.0739
0.2	256.791	256.613	0.178	0.0694
0.3	119.611	119.535	0.076	0.0636
0.4	71.598	71.558	0.040	0.0559
0.5	49.375	49.351	0.024	0.0486
0.6	37.304	37.288	0.016	0.0429
0.7	30.026	30.014	0.012	0.0400
0.8	25.302	25.293	0.009	0.0356
0.9	22.064	22.056	0.008	0.0363
1.0	19.748	19.741	0.007	0.0354
1.1	18.034	18.028	0.006	0.0333
1.2	16.731	16.725	0.006	0.0358
1.3	15.717	15.711	0.006	0.0381
1.4	14.912	14.906	0.006	0.0403
1.5	14.263	14.257	0.006	0.0421
1.6	13.732	13.726	0.006	0.0437
1.7	13.291	13.286	0.005	0.0376
1.8	12.923	12.917	0.006	0.0464
1.9	12.610	12.604	0.006	0.0476
2.0	12.344	12.338	0.006	0.0486

Table 2. Values of natural frequency, λ, for $\varphi_1 = \varphi_3 = 1$, $\varphi_2 = 0.5$, $p = b/a$

p	λ (radians per second)		Difference	Percentage difference
	Present study	Pilkey (2005)		
0.1	992.675	991.934	0.741	0.0747
0.2	252.004	251.822	0.122	0.0723
0.3	114.996	114.917	0.079	0.0687
0.4	67.211	67.168	0.043	0.0640
0.5	45.257	45.230	0.027	0.0597
0.6	33.481	33.462	0.019	0.0568
0.7	26.509	26.495	0.014	0.0528
0.8	22.090	22.079	0.011	0.0498
0.9	19.145	19.136	0.009	0.0470
1.0	17.104	17.096	0.008	0.0468
1.1	15.643	15.636	0.007	0.0448
1.2	14.569	14.562	0.007	0.0481
1.3	13.761	13.754	0.007	0.0509
1.4	13.140	13.133	0.007	0.0533
1.5	12.654	12.648	0.006	0.0474
1.6	12.268	12.261	0.007	0.0571
1.7	11.956	11.949	0.007	0.0586
1.8	11.701	11.695	0.006	0.0513
1.9	11.491	11.484	0.007	0.0610
2.0	11.314	11.307	0.007	0.0619

Table 3. Values of natural frequency, λ, for $\varphi_1 = 1$, $\varphi_2 = \varphi_3 = 0.5$, $p = b/a$

p	λ (radians per second)		Difference	Percentage difference
	Present study	Pilkey (2005)		
0.1	705.423	704.902	0.521	0.0739
0.2	181.713	181.587	0.126	0.0694
0.3	84.865	84.812	0.053	0.0625
0.4	51.107	51.078	0.029	0.0568
0.5	35.605	35.587	0.018	0.0506
0.6	27.287	27.275	0.012	0.0440
0.7	22.351	22.341	0.010	0.0448
0.8	19.206	19.198	0.008	0.0417
0.9	17.094	17.086	0.008	0.0468
1.0	15.613	15.606	0.007	0.0449
1.1	14.539	14.533	0.006	0.0413
1.2	13.738	13.732	0.006	0.0437
1.3	13.126	13.120	0.006	0.0457
1.4	12.648	12.641	0.007	0.0554
1.5	12.268	12.261	0.007	0.0571
1.6	11.961	11.954	0.007	0.0586
1.7	11.709	11.703	0.006	0.0513
1.8	11.501	11.495	0.006	0.0522
1.9	11.326	11.320	0.006	0.0530
2.0	11.179	11.172	0.007	0.0627

Conclusion

The closeness of the results obtained from the formulated natural frequency equation to those of Pilkey (2005), shows that the solution obtained in the present study is a good approximation of the exact solution. Therefore, Taylor's series is adequate for the derivation of the deflection function for all-round simply supported rectangular orthotropic thin plate subject to free vibration. The application of Taylor's series in plate vibration problems will overcome the deficiencies of trigonometric series and may be extended to other boundary conditions whose deflection functions cannot be formulated using trigonometric series functions. The solutions obtained using Taylor's series deflection functions are obtained with less computation efforts.

References

Abamara, A.C. (2014). *Free vibration analysis of material orthotropic rectangular thin plates using Taylor-Maclaurin series in Rayleigh-Ritz method*, (Unpublished master's thesis). Federal University of Technology, Owerri, Nigeria.

Hsu, M. (2003). Vibration analysis of isotropic and orthotropic plates with mixed boundary conditions. *Tamkang Journal of Science and Engineering*, ***6***(4), 217-226.

Ibearugbulem, O.M. (2012). *Application of direct variational principles in elastic stability analysis of thin rectangular isotropic plates*, (Unpublished doctoral dissertation). Federal University of Technology, Owerri, Nigeria.

Leissa, A. W. (1973). The free vibration of rectangular plates. *Journal of Sound and Vibration*, ***31***(3), 257-293.

Meirovitch, L. (1997). *Principles and techniques of vibration*. New Jersey: Prentice-Hall.

Pilkey, W.D. (2005). *Formula for stresses, strains, and structural matrices* (2nd ed.). New Jersey, NJ: John Wiley and Sons.

Srinivas, S. & Rao, A.K. (1970). Bending, vibration, and buckling of simply supported thick orthotropic rectangular plates and laminates. *International Journal of Solids and Structures*, ***6***, 1463-1481.

Srinivas, S., Rao, C.V.J. & Rao, A.K. (1970). An exact analysis for vibration of simply-supported homogenous and laminated thick rectangular plates. *Journal of Sound and Vibration*, ***12***, 187-199.

Ugural, A.C. (1999). *Stresses in plates and shells* (2nd ed.). Singapore: McGraw-Hill.

Ventsel, E. & Krauthammer, T. (2001). *Thin plates and shells: Theory, analysis, and applications*. New York, NY: Marcel Dekker.

Wu, J.H., Liu, A.Q. & Chen, H.L. (2007). Exact solutions for free-vibration analysis of rectangular plates using Bessel functions. *ASME Journal of Applied Mechanics*, ***74***, 1247-1251. DOI: 10.1115/1.2744043

Xing, Y.F. & Liu, B. (2009). New exact solutions for free vibrations of thin orthotropic rectangular plates. *Composite Structures*, *89*, 567-574. DOI:10.1016/j.compstruct.2008.11.010